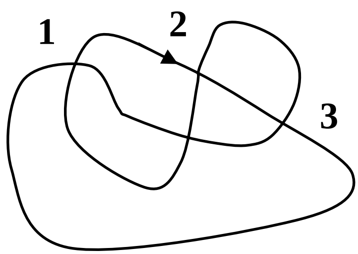
12. ENUMERATING KNOTS

§12.1. Scribbles

A **scribble** is a closed path in the plane such that every crossing is a single crossing, as in a knot map. So every knot map is a scribble, and vice versa. The only reason for using both terms is that 'scribble' removes all connections of the path with knots. So a 2-dimensional mathematician would understand scribbles, but not knot maps.

Example 1:

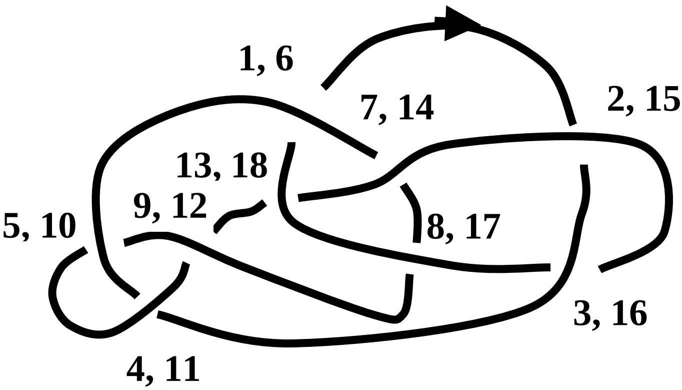


Here we get back to vertex 1 before the final vertex has received its label. Of course we can jump over the 1 and label the remaining crossing as 4, but this doesn't seem all that systematic. A better system is to label every second crossing. This will ensure that every crossing gets labelled before all of them have been labelled.

Theorem 1: If the n crossings of a scribble are labelled in order, starting at an arbitrary crossing and proceeding around one of the two possible directions: 1, 2, 3, … , $2n$

then each vertex will receive two labels of opposite parity (one will be odd and the other will be even).

Proof: If the two labels at a crossing C had the same parity the portion of the knot between these two visits to C will be a closed path cut by the other portion of the knot in an odd number of places, a contradiction (the rest of the knot must enter and exit this closed region an equal number of times).



So if we label every *second* crossing we will allocate the crossing whose odd label is n to ½(n – 1).

§12.2. Scribbles and Permutations

If we label the vertices of a scribble, S, with n crossings as described above the scribble can be described by a permutation. Start again at crossing 1 and proceed around the knot, in the same direction that we chose for the labelling and record the labels of the crossings as we proceed around the scribble, until we return to crossing 1. The list will be of the form $1, x_1, 2, x_2, \ldots, n, x_n, 1$

We can write this in a table:

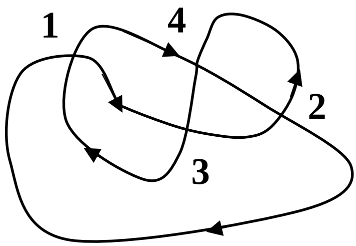
1	2	…	n
x_1	x_2	…	x_n

This can be interpreted as a permutation π(S), which we can write in cycle notation.

Example 2: For this scribble the permutation is

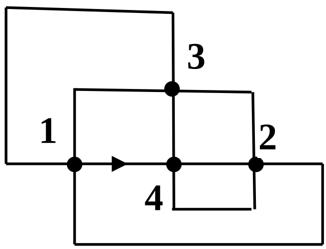
1	**2**	**3**	**4**
4	**1**	**2**	**3**

which can be written as (1432).



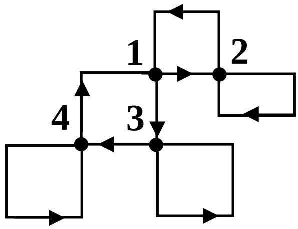
Given a permutation we can attempt to reconstruct the scribble.

Example 3: Find a scribble S for which π(S) = (1432).
Solution:



Example 4: Draw a scribble S for which $\pi(S) = (12)$.
Solution: We write the permutations as

1	**2**	**3**	**4**
2	**1**	**3**	**4**



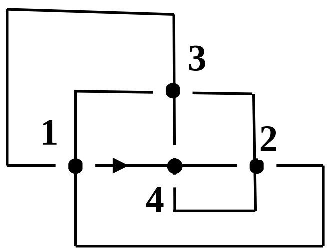
§12.3. Knots and Permutations

A knot projection is a scribble with the additional information about overpasses and underpasses. If these alternate then we can infer this information by making an assumption about the first crossing.

An **alternating knot** is one where underpasses and overpasses alternate. Most knots, when represented by a projection with the least number of crossings, are alternating — but not all.

So a permutation describes an alternating knot. Well, actually it describes a conjugate pair because there is the choice of the under- and over-passes at the first crossing.

Example 5: The permutation (13)(24) gives us the Figure 8 knot.

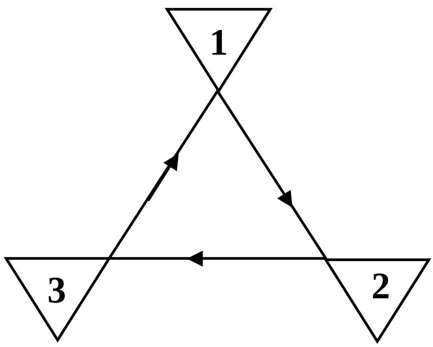


Making the opposite choice of crossings gives the same knot because the Figure 8 Knot is amphicheiral.

Example 6: Find all the knots with crossing number equal to 3.

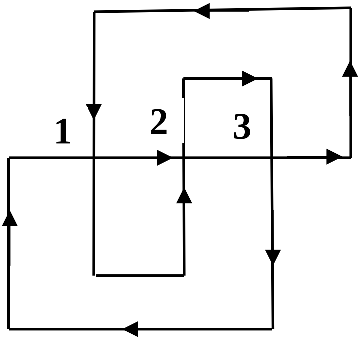
Solution: The unknot has crossing number 0. It should be obvious that there are no knots whose crossing number is 1 or 2. There are 6 permutations in $\mathbf{S}_3$: I, (123), (132), (12), (13) and (23). Clearly if a permutation fixes a symbol the scribble will have a 'kink' which could be removed by a Type I Reidemeister Move. So to get a knot with crossing number 3 we can only use the permutations (123) and (132).

Now (123) would result in the crossing sequence 1 2 2 3 3 1 1 and hence the scribble:



Clearly, no matter how we arranged the over- and under-crossings this would result in the unknot.

The permutation (132) results in the crossing sequence: 1321321 which results in the scribble:



The alternating knots are the conjugate Trefoil knots. As mentioned earlier they are not equivalent. It should also be obvious that any non-alternating pattern of over- and under- crossings would result in an unknot.

§12.4. Knots With Crossing Number Up to 4.

So far our catalogue of knots consists of the following:

Crossing Number	Knot	Alexander No.
0	Unknot	1
1	None	
2	None	
3	Trefoils	3

We know of the Figure 8 knot, which has crossing number 4. Are there any others?

Example 7: Find all the knots with crossing number equal to 4.

Solution: There are 24 permutations on 4 symbols, but we don't get nearly that many alternating knots whose crossing number is 4. The 3-cycles and 2-cycles fix a symbol and so the corresponding knots would have a 'kink' which could be removed by a Type I Reidemeister Move, giving a knot with a lower crossing number. Moreover any permutation, π, where $\pi(1) = 2$, $\pi(2) = 3$, $\pi(3) = 4$ or $\pi(4) = 1$, would also result in a removable 'kink'.

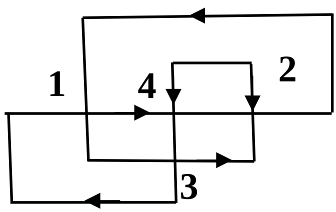
There are six 4-cycles and three double 2-cycles. This table lists them, together with the corresponding tables.

(1234)	**(1243)**	**(1324)**	**(1342)**	**(1423)**	**(1432)**
1 2 3 4 2 3 4 1	1 2 3 4 2 4 1 3	1 2 3 4 3 4 2 1	1 2 3 4 3 1 4 2	1 2 3 4 4 3 1 2	1 2 3 4 4 1 2 3

The first of these has the removable 'kink' 2-2 and so must be rejected. So has the second. In the case of the third there's a removable 'kink' 1-1. In fact the only one that survives is the last, which results in the crossing list

1-4-2-1-3-2-4-3-1

Can we draw this as a scribble? Yes.



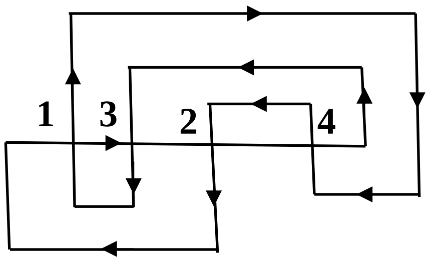
Now we consider the double 2-cycles. Here they are, together with the corresponding tables.

(12)(34)	**(13)(24)**	**(14)(23)**
1 2 3 4 2 1 4 3	1 2 3 4 3 4 1 2	1 2 3 4 4 3 2 1

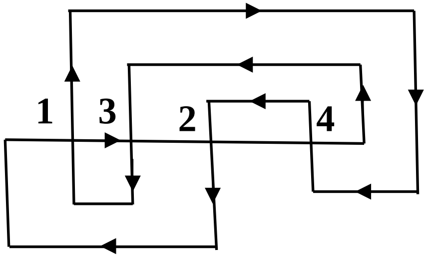
Only the middle one has no removable ‘kinks’.

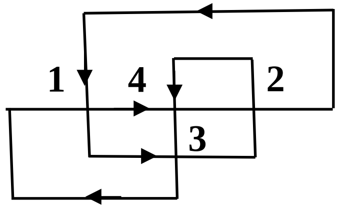
(13)(24)
1 2 3 4 3 4 1 2

Can we draw a corresponding scribble? Yes.

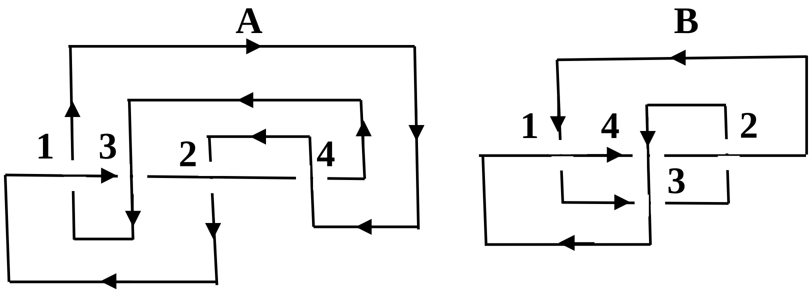


We have two scribbles that might become knots:





Choosing one of the two possibilities for the over- and under-passes at crossing 1, we have the following:



The Alexander Numbers of these knots are 5 and 1, respectively. This suggests that A is the Figure 8 Knot and B is the trivial knot. However, to prove these assertions we need to actually deform A to the standard picture of the Figure Eight Knot and B to an unknotted circle, either with a string model or as a series of pictures. With B it's easy to see that a Reidemeister Type II move will reduce the number of crossings. So our conclusion is that there is only one conjugate pair with crossing number equal to 4. The fact that the Figure Eight Knot is amphicheiral means that, in fact, there is only one knot whose crossing number is 4.

When it comes to non-alternating knots we have the extra complexity of considering the nature of the crossings. With a fair amount of work we can show that there are no non-alternating knots with crossing number 4. Among the possibilities of knots with 4 crossings we will find one that reduces to the (alternating) Trefoil knots. In fact there are no non-alternating knots whose crossing number is 7 or less.

This analysis should give you some idea how one goes about compiling a catalogue of knots. We build it up in stages, based on the crossing number.

For each n, we list all the permutations, π, in $\mathbf{S}_n$ such that for all i, $\pi(i) \neq i$ and $\pi(i) \neq i + 1 \pmod{n}$. For each of these permutations we attempt to draw a corresponding scribble. This won't always be possible because we may sometimes find ourselves on one side of a closed loop, having to reach a crossing on the other side. (This didn't happen for $n = 4$, but it will for larger n.

For alternating knots we choose the nature of crossing 1 and turn the scribbles into knots. Then we compare these with the knots with smaller crossing number, which we've already listed. Any knot with that has already been included under a smaller crossing number will need to be eliminated.

Finally we have to consider the possibility that some of the knots that remain are repetitions of the same knot, arising from different permutations.

This will give us all conjugate pairs of alternating knots. The next job will be to determine which of our knots are amphicheiral, which is very difficult indeed. Finally, we need to consider non-alternating knots. Here each permutation will give rise to many possible knots, but the vast majority will reduce to a knot with fewer crossings.

One of the exercises will ask you to extend the Knot Catalogue to include all those with crossing number 5. An appendix contains a list of all knots whose crossing number is 7 or less. There are two knots with crossing number 5, six with crossing number 6 and nine with crossing number 7. All of them are alternating.

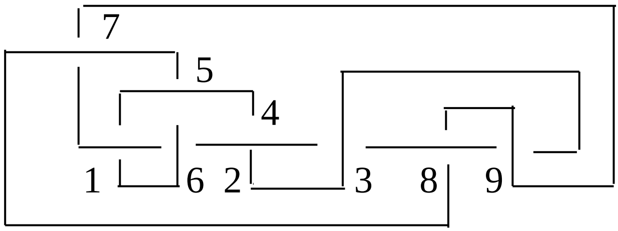
EXERCISES FOR CHAPTER 12

Exercise 1: Draw the alternating knot described by the permutation: (165)(24)(387). Label the vertices using the method shown in this chapter.

Exercise 2: Find all the knots with crossing number 5. You may assume that they are all alternating.

SOLUTIONS FOR CHAPTER 12

Exercise 1:



Exercise 2:

The cycle structures of the permutations in $\mathbf{S}_5$ that don't fix a symbol are (×××××) and (×××)(××).

There are 24 cycles of length 5, but the only ones where $\pi(i) \neq i + 1 (\text{mod } 5)$ are (14253), (14352), (13524), (13542), (14352), (15243) and (13254).

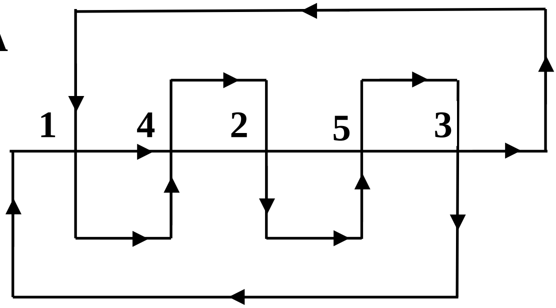
The tables for these are:

A				
(14253)	**(14352)**	**(13524)**	**(13542)**	**(13254)**
1 2 3 4 5 4 5 1 2 3	1 2 3 4 5 4 1 5 3 2	1 2 3 4 5 3 4 5 1 2	1 2 3 4 5 3 1 5 2 4	1 2 3 4 5 3 5 2 1 4

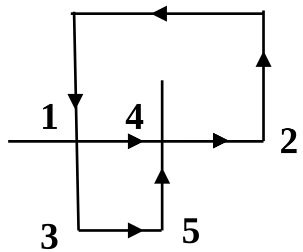
(15243)	**(15324)**
1 2 3 4 5 5 4 1 3 2	1 2 3 4 5 5 4 2 1 3

The first of these produces a list of crossings: 14253142531. This leads to a scribble.

A



The second of these fails to give a scribble. The crossing list is 14213543521.



There is no way that this can lead to a scribble, without introducing additional crossings. None of the other 5-cycles give a valid scribble.

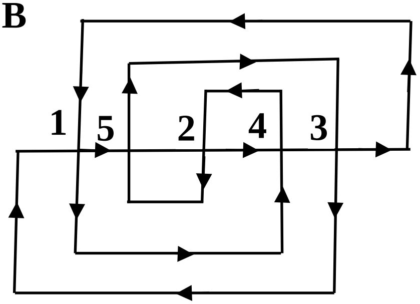
So far we just have the scribble A. Before we turn it into a knot, let's see what we get from the permutations of the form (×××)(××).

There are 20 of these. But, imposing the restriction that $\pi(i) \neq i + 1 (\mathrm{mod}\ 5)$ we are left with just the following: (153)(24), (142)(35), (143)(25) and (253)(14). The corresponding crossing tables are:

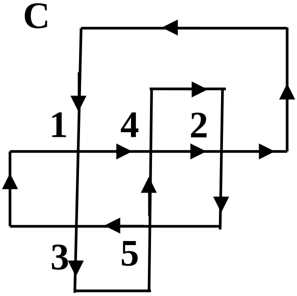
B	C	D	E
(153)(24)	**(142)(35)**	**(143)(25)**	**(253)(14)**
1 2 3 4 5 5 4 1 2 3	1 2 3 4 5 4 1 5 2 3	1 2 3 4 5 4 5 1 3 2	1 2 3 4 5 4 5 2 1 3

All of these give rise to valid scribbles.

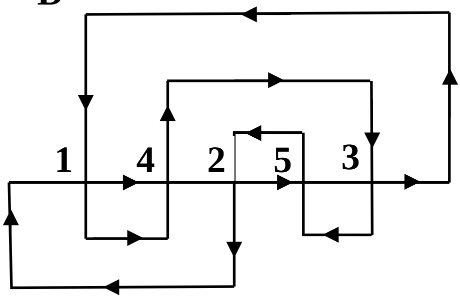
B



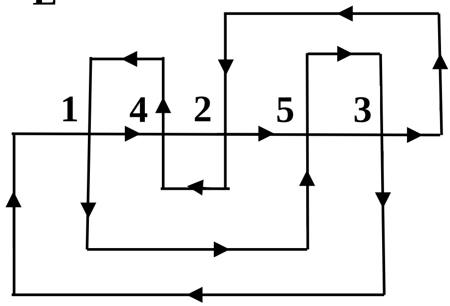
C



D

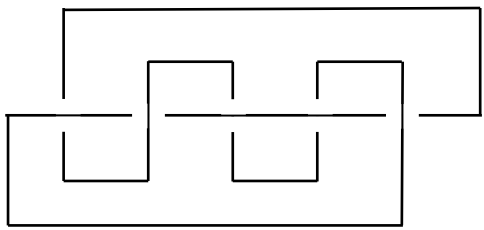


E

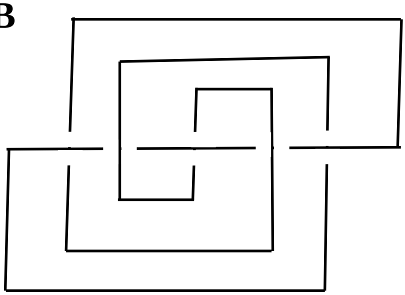


The alternating knots that correspond to these scribbles are:

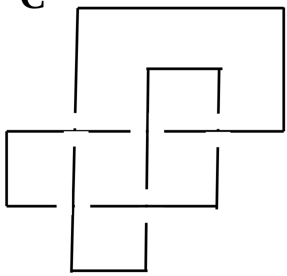
A



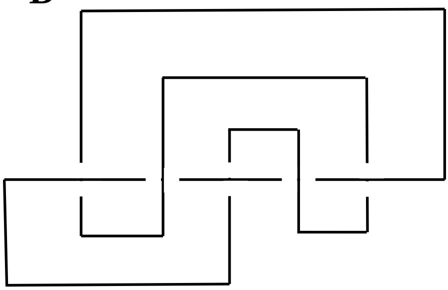
B



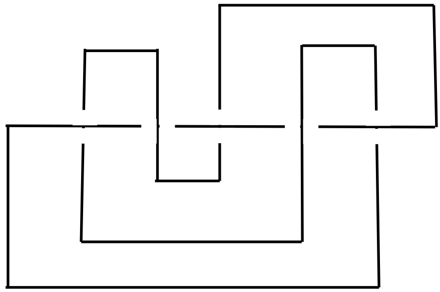
C



D



E



The Alexander Numbers of these knots are as follows:

A	B	C	D	E
5	7	7	7	7

Consulting our list of knots with crossing number 3 or 4, we conclude that all these knots must have crossing number 5. But are they all different? Knot A certainly differs from the rest. But perhaps B to E are all the same. In fact they are, but the only way to demonstrate this is to use a piece of string, or a series of Reidemeister diagrams.

Using the notation in the Appendix, we see that knot A is 5.1 and knots B to E are copies of knot 5.2.